An Internal Ballistics System Based on an A Priori Derivation of the Leduc Equation Constants

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1. Introduction

The Leduc internal ballistics system was first published in 1904 in an article by Challeat [1] on the 'Theory of Recoil', with attribution to Leduc. The system depends on assuming a simple hyperbolic relation between shot velocity and distance travelled up the barrel, which closely corresponds to the form determined both theoretically and experimentally. The eponymous equation is,

$$v = \frac{ax}{b+x} \tag{1}$$

where v is shot velocity, x is the distance of travel up the barrel, and 'a' and 'b' are constants. The derivative of Eq. (1) also closely corresponds to the form for pressure against shot travel.

Challeat gave equations for the two constants expressed in terms of the propellant properties and loading conditions, but they were derived with little support from thermodynamic theory and were described as "fallacious" by Hunt [2]. They were modified by Alger [3] for use with US Navy propellants, for which they were used until 1942. Serabryakov [4] modified them for Soviet propellants. However, they have never proved to be of much use in a general sense.

Nevertheless, the Leduc equation can be made to agree with the experimental values very well for a given gun, once the two constants have been determined, as has been demonstrated for large guns [5] and for small arms (see, for example [6]). In consequence, the Leduc formula for velocity as a function of distance travelled became a mainstay in the determination of some internal ballistic properties in the United States right up until the 1970s [7]. Today, the Leduc equation is probably the best known internal ballistics equation and is often cited in the popular media.

It is, however, possible to derive the two constants of the Leduc equation in terms of the loading conditions and propellant properties, based on sound thermodynamic principles, provided certain assumptions are made. The assumptions are as follows.

2. Basic Assumptions

2.1 The shot start pressure is zero.

That is, there is no separate variable for shot start pressure and it is assumed that the shot starts to move immediately the propellant starts to burn. Shot start pressure is a significant factor only where it is not small compared to the maximum breech pressure, or for handguns and small rifle cartridges where very fast propellants are used and the time taken for the projectile to engrave the rifling is of the same order as rise time to maximum pressure.

2.2 The covolume of the propellant gasses is equal to the reciprocal of the original charge density.

This approximation is of no consequence at the start of the projectile's journey down the barrel, when the pressure is still rising quickly to peak pressure and relatively little of the propellant has burnt. When the projectile is well advanced down the barrel, the expansion ratio is such that the system has effectively lost memory of the starting conditions anyway, so this is not a serious impediment.

2.3 The burning rate of the propellant is linearly proportional to the pressure.

This has been found to be a good approximation in practice for the pressures at which guns generally operate, and is in any case a common assumption even in sophisticated numerical internal ballistics systems.

2.4 The area of the burning surface remains constant.

This would seem to be a reasonable assumption in practice. This assumption is approximately true for "neutral" propellant kernel forms, such as flakes and cylindrical propellants with one perforation, which are probably the most common kernel shapes in small-arms propellants. Spherical ball propellants have a naturally digressive shape, but this is usually mitigated by deterrent coatings which make such propellants burn in more of a neutral manner. It is well known that progressive forms such as cylinders with multiple perforations tend to splinter relatively early in their burning cycle compared to simpler forms, so that they actually burn in a more neutral manner than their form would suggest.

2.5 That no energy is lost through friction as the shot travels up the barrel.

There is no separate variable for friction. This is commonly accounted for by increasing the effective mass of the shot.

2.6 That no energy is lost through heat to the walls of the chamber or barrel.

There is no separate variable for heat loss. This is commonly accounted for by increasing the value of gamma, the ratio of specific heats, in order to mimic heat losses.

2.6 That the propellant is all burnt within the gun barrel.

This is, in any case, a desirable condition

3. Determination of the Leduc Equation Constants

3.1 The Constant 'a'

If a barrel of infinite length is assumed, it is obvious from Eq. (1) that as x, the distance travelled in the barrel, becomes very large, then $v \rightarrow a$ and so the constant a is the asymptotic value of the shot velocity as $x \to \infty$. Following Hunt [2], Rèsal's equation is invoked,

$$P_{\mathrm{M}} V \;=\; F\,C \;-\; (\gamma - 1)\frac{m\,v^2}{2} \tag{2}$$

where P_{M} is the mean gas pressure in the barrel behind the shot, the pressure at which the propellant is said to burn. *V* is the free volume behind the shot, *m* is the effective mass of the shot, *F* is the 'Force' of the propellant, *C* is the charge weight of propellant and γ is the ratio of specific heats for the propellant gasses. At $x = \infty$ the pressure will be zero and Eq. (2) can be rearranged so that,

$$v \;=\; \left[\frac{2FC}{m(\gamma - 1)}\right]^{\frac{1}{2}} \;=\; a \tag{3}$$

3.2 The Constant 'b'

Following Challeat, the acceleration of the shot up the barrel can be determined by differentiating Eq. (1)

$$\frac{dv}{dt} \;=\; \frac{dv}{dx}\frac{dx}{dt} \;=\; v\frac{dv}{dx} \;=\; \frac{a^2\,b\,x}{(b + x)^3} \tag{4}$$

Maximum pressure will occur when the rate of change of acceleration is zero,

$$\frac{d^2v}{dt^2} \;=\; \frac{a\,x(a^2\,b^2 - \;2\,a^2bx)}{(b\; + x)^5} \;=\; 0 \tag{5}$$

which will be true when,

$$x_{\text{P-Max}} \;=\; \frac{b}{2} \tag{6}$$

It follows that the shot velocity at maximum pressure will be,

$$v_{\text{P-Max}} \;=\; \frac{a}{3} \tag{7}$$

The shot base pressure P_{S} acting on the base of the shot at any point in the barrel can then be written as,

$$P_{\mathrm{S}} = \frac{m}{A}\frac{dv}{dt} \;=\; \frac{m\,a^2b\,x}{A(b + x)^3} \tag{8}$$

where *A* is the area of the bore.

The mean gas pressure can be related to the shot base pressure by the well known Lagrange pressure relationship (see, for example [8]) such that,

$$\frac{P_{\mathrm{M}}}{P_{\mathrm{S}}} \;=\; \left(1 \;+\; \frac{C}{3\,m_{\mathrm{S}}}\right) \;=\; \Phi \tag{9}$$

where m_S is the shot weight. The mean gas pressure can then be expressed as,

$$P_M = \frac{\Phi m a^2 b x}{A(b+x)^3} \tag{10}$$

The maximum mean gas pressure $P_{P\text{-Max}}$ will be when the shot has travelled a distance $b/2$ so that,

$$P_{P\text{-Max}} = \frac{4\Phi m a^2}{27 b A} \tag{11}$$

However, the mean gas pressure can also be expressed in terms of Rèsal's equation, so the maximum mean gas pressure can be written as,

$$P_{P\text{-Max}} = \frac{F C Z - (\gamma - 1)\dfrac{m v^2}{2}}{\left(V_0 + \dfrac{b A}{2}\right)} \tag{12}$$

Z is fractional amount of the propellant charge C that has burnt, ($Z = 0$ as the propellant starts to burn, $Z = 1$ when the charge is all burnt). V_0 is the free volume behind the loaded shot where,

$$V_0 = \left(V_C - \frac{C}{\rho}\right) \tag{13}$$

V_C is the chamber capacity (or usable case capacity where a case is used) and ρ is the propellant density (not the loading density).

Note that from Eq. (8), the velocity can be expressed as,

$$v = \frac{A}{m\Phi}\int P_M \, dt \tag{14}$$

From Vielle's Law, the rate at which the charge C is burnt can be expressed as (see, for example [9]),

$$\frac{dZ}{dt} = \frac{2\beta P_M}{w} \tag{15}$$

Where w is the characteristic web thickness of the propellant kernel and β is the linear burning rate coefficient of the propellant. (Note here that the burning rate is used in the American sense as the rate of regression of the propellant kernel surface, hence the factor 2.) Then integrating Eq. (15),

$$Z = \frac{2\beta}{w} \int P_{\mathrm{M}}\, dt \tag{16}$$

Combining Eq. (14) and Eq. (16),

$$Z = \frac{2\beta m \Phi v}{A w} \tag{17}$$

Substituting then for Z in Eq. (12), and for v from Eq. (7),

$$P_{\text{P-Max}} = \frac{\dfrac{2FC\beta m\Phi a}{3Aw} - (\gamma - 1)\dfrac{m a^2}{18}}{\left(V_0 + \dfrac{bA}{2}\right)} \tag{18}$$

$P_{\text{P-Max}}$ can now be substituted from Eq. (11), and the resultant equation solved for 'b', so that,

$$b = \frac{4aV_0}{\left[\dfrac{18FC\beta}{w} - aA\left(2 + \dfrac{3(\gamma - 1)}{2\Phi}\right)\right]} \tag{19}$$

The constants 'a' and 'b' in the Leduc equation have now been rendered in terms of the propellant properties and the loading conditions on a sound thermodynamic basis. The constants can thus serve as ballistic coefficients in an internal ballistics system.

4. 'All-burnt'

'All-burnt' happens when $Z = 1$. The shot velocity at all-burnt can be determined from Eq. (17), and so the position of all-burnt from Eq. (1). Thus, all-burnt happens at,

$$x_{\text{all-burnt}} = \frac{b}{\left(\dfrac{2\beta m\Phi a}{Aw} - 1\right)} \tag{20}$$

It is usual in analytic systems to assume adiabatic expansion after all-burnt, but since the primary equation for velocity as a function of distance travelled in the barrel is given by the Leduc equation, Eq. (1), there is no need to make such an assumption in this system.

5. Pressure as a Function of Time

Chamber or breech pressure as a function of time is almost always a measured parameter to determine the internal ballistics performance of a gun, so it is convenient if a form for pressure vs time can be derived. An obvious route is a function for distance travelled vs time, from which pressure vs time could be easily obtained via Eq. (10).

Following Challeat once more,

$$dt = \frac{dx}{v} = \frac{(b + x)}{ax}dx \tag{21}$$

Integrating between the limits of t_1 to t and from x_1 to x,

$$t = t_1 + \frac{1}{a}\left[b\ln\left(\frac{x}{x_1}\right) + x - x_1 \right] \tag{22}$$

But note that there is a singularity at $x_1 = 0$, so this expression cannot be used to determine the entire time history of the pressure. Struble [10] showed that it is actually not possible to determine an expression for pressure vs time when it is assumed that the burning rate is proportional to pressure.

Challeat noted that if x_i was set to $b/2$, the shot travel distance to maximum pressure, then Eq. (22) could be used to give time as a function of shot travel (and so pressure) after the point of maximum pressure. Challeat, and later Serabryakov [11], gave expressions for the time up to maximum pressure assuming uniform shot acceleration up to maximum pressure, but these estimates are generally too small by about a factor of two when compared to more accurately determined times.

However, where both pressure and velocity can be expressed as a function of distance travelled, it is always possible to make a complete determination of pressure as a function of time by plotting pressure at distance *x* vs *t(x)* where,

$$t(x) = \sum_{x'=0}^{x'=x} \frac{2\,\Delta x'}{v(x') + v(x'+\Delta x')} \tag{23}$$

Here, $\Delta x'$ is some small interval in x such that the rate of change in pressure in any interval from x' to $x'+\Delta x'$ is approximately uniform.

6. Vivacity

Today, it is common for propellant manufacturers to express the burning rate characteristics of their propellants in terms of vivacity. Vivacity is the fractional rate at which the original propellant mass burns away per unit of pressure and is usually plotted as a function of Z, the amount of propellant that has burnt. Thus Eq. (15) may be expressed as,

$$\frac{dZ}{dt} = \frac{2\beta}{w} P_{\mathrm{M}} = \Lambda P_{\mathrm{M}} \tag{24}$$

where Λ is the vivacity.

The vivacity used in this system would be an averaged vivacity for the propellant. Practically, this should be determined for the vivacity between $Z = 0$ and $Z \approx 0.6$, after which, the vivacity usually falls away rapidly to zero. But this later part of the burning cycle has no effect on the maximum pressure and little effect on the muzzle velocity.

Note that the web thickness has now dropped out of the equation for Z and so there is no dependence on kernel geometry or dimensions when vivacity is used.

7. Accounting for Losses

7.1 Propellant kinetic energy and losses due to friction

It is common to use the approximation of Lagrange and suppose that the kinetic energy of the propellant and gasses is equivalent to one third of their combined weight moving at the same velocity as the shot (see, for example [12]).

It is also common to suppose that an amount equivalent to about 1% of the shot kinetic energy is lost due to recoil, and about 4% due to shot friction in the barrel (see, for example [8]).

Thus the effective mass will then be,

$$m = 1.05\, m_{\mathrm{S}} + \frac{C}{3} \tag{25}$$

7.2 Accounting for heat loss to the barrel

By assuming that the heat loss is proportional to the shot kinetic energy, and so to v^2, it is possible to account for heat loss by making an upward adjustment to gamma, the ratio of specific heats. This mimics the heat lost to the barrel by reducing the thermodynamic efficiency of the system. Gamma as used above may be redefined as an effective gamma using (see, for example [13]),

$$(\gamma - 1) = (\gamma_{\mathrm{P}} - 1)(1 + \psi) \tag{26}$$

where γ_{P} is the measured (or estimated) ratio of specific heats for the propellant and ψ is the heat loss as a fraction of the kinetic energy of the shot. For small arms, experience would suggest that $\psi \approx 1$. For medium size guns $\psi \approx 0.6$ and for large guns $\psi \approx 0.17$.

8. Breech Pressure

The chamber or breech pressure P_B, as measured by a piezo transducer, can be related to the mean pressure of the gasses behind the shot by the Lagrange relation (see, for example [8]),

$$\frac{P_B}{P_M} = \left(\frac{1 + \dfrac{C}{2m_S}}{1 + \dfrac{C}{3m_S}} \right) \tag{27}$$

Expressions above for mean gas pressure may be related to the breech pressure using Eq. (27).

9. An Example

As an example, the system described above is used to predict the interior ballistics of type M80 7.62 x 51 NATO ammunition, as fired in an M14 rifle. The specifications are listed in Tab. 1.

Tab. 1 Typical specification of M80 7.62 x 51 ball ammunition in an M14 rifle

Muzzle velocity	845 m/s
Shot weight	9.5 gm
Maximum chamber pressure	345 Mpa
Usable case capacity	3207 mm^3
Charge weight	2.94 gm
Barrel length	558 mm
Chamber length	51 mm
Distance from shot base to muzzle	515 mm
Bore area	47.51 mm^2

A propellant commonly used was type WC846 which had a rolled-ball kernel form. Its initial burning properties were modified by deterrents. The relevant properties [14] are listed in Tab. 2.

Tab. 2 Typical properties of WC846 propellant

Average kernel thickness	0.381 mm
Average kernel diameter	0.5473 mm
Average kernel volume	0.067 mm^3
Ratio of specific heats (estimated average)	1.25
Force (estimated average)	1005 J/gm
Burning rate (average of various estimates)	7.57 x 10^{-7} mm/s/ Pa
Propellant density	1.62 gm/cm^3

Using the given thickness and diameter for the rolled-ball kernel, an initial surface area of 0.600 mm^2 can be determined. An equivalent hypothetical flake kernel can be envisaged, where the area of the top and bottom is each 0.300 mm^2, and where the thickness is 0.223 mm, such that the volume of this flake kernel is the same as the rolled-ball kernel given in Tab. 2. If it is assumed that this hypothetical flake kernel only burns on the top and bottom and not on the sides, then its rate of burning will be constant. Let the characteristic web thickness w then be 0.223 mm.

Tab. 3 Calculated results using the parameters from Tab. 1 and Tab. 2 as inputs

Value of constant 'a' from Eq. (3)	1039 m/s
Value of constant 'b' from Eq. (19)	0.120 m
Muzzle velocity from Eq. (1)	842 m/s
Maximum breech pressure from Eq. (11) and Eq. (27)	354 MPa
Velocity at maximum pressure from Eq. (7)	346 m/s
Distance travelled to maximum pressure from Eq. (6)	0.060 m
Distance travelled to all-burnt from Eq. (20)	0.194 m

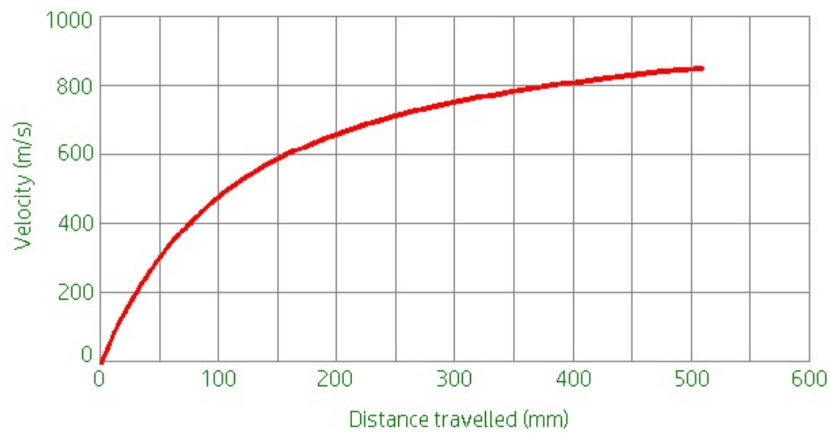

Fig. 1 Leduc curve for velocity vs distance travelled in the barrel from Eq. (1)

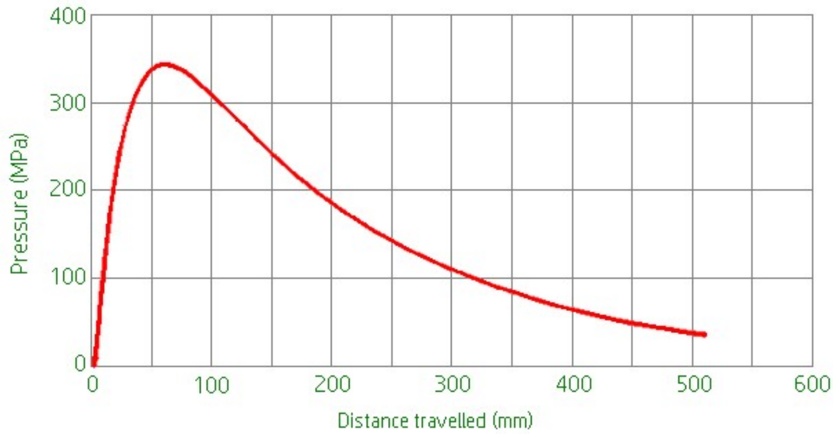

Fig. 2 Breech pressure vs distance travelled from Eq. (10) and Eq. (27)

Fig. 3 shows the theoretical breech pressure as a function of time, as generated using data from the curves in Fig. 1 and Fig. 2, according the procedure set out in Section 5. This is compared to a typical measured in-chamber pressure vs time curve for the M80 cartridge using the WC846 propellant [15]. As can be seen, the agreement is very reasonable.

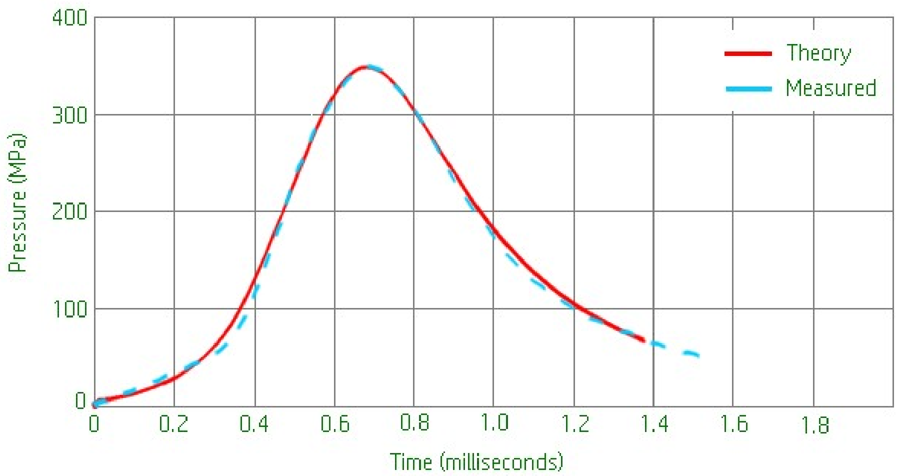

Fig. 3 Theoretical curve for breech pressure vs time, compared to a measured curve

10. Discussion

Muzzle velocities determined using this system would seem to be in good general agreement with those predicted by the Mayer-Hart system [16], which shares the basic assumptions listed in Section 2. The Mayer-Hart system was considered by the US Ballistics Research Laboratory to be a reliable predictor of muzzle velocities across a wide range of gun types with an average error of under 2% [7] .

Given the assumption that the burning rate is constant, it would not be expected that this system would be an accurate predictor of maximum pressure. Maximum pressure is very sensitive to whether, and to what extent, the propellant burns in a progressive, digressive, or neutral manner during the first 20% or so of its burning cycle. For this reason, maximum breech pressures calculated using this system should probably be considered as indicative rather than predictive.

Whilst the predictions of distance travelled to all-burnt will generally not be unrealistic, they will usually be underestimated due to the assumption that the rate of burning is constant. In reality, splintering of the propellant kernels starts at $Z \approx 0.5$, after which the propellant burns in a more and more digressive manner, such that it takes a comparatively long time for final elements of the propellant to burn.

11.Conclusions

The constants for the Leduc equation have been rendered in terms of propellant properties and loading conditions on the basis of sound thermodynamic theory, whilst making certain assumptions which are common in most classical analytic internal ballistics systems [17] and in many zero dimensional numerical systems. Using these constants as ballistics coefficients, a simple internal ballistics system of equations has been described.

Even today, where sophisticated numerical systems are widely available, there is a place for analytic systems which can be used as a rapid 'reality check' when assessing a numerical system, or to give reasonable estimates of various basic ballistic parameters where extreme accuracy is not required.

This system is surprisingly accurate given that the principal equations are probably the simplest of any a priori analytic internal ballistics system yet devised.

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