An Analytical System for Internal Ballistics

Introduction

After a good understanding of thermodynamics and thermochemistry was achieved in the late 19th century, and before the advent of fast computing power in the middle of the 20th century, a great deal of ingenuity was expended on developing closed mathematical functions to describe the progress of the projectile (principally artillery shells) down the barrel of the gun.

From the beginning however, there were two objectives for the attempted solutions, which were to some extent diametrically opposed. In one camp were the “mathematical ballisticians”, who sought general “exact” solutions to the basic internal ballistic equations in order to achieve a proper understanding of the inter-relationship of these equations and their various parameters. In the other camp were the “practical ballisticians”, who were prepared to admit approximations and simplifications to achieve working solutions that were accurate enough to be useful, but requiring the least amount of computational effort.

In the first camp was (for example) the work of Kapur [1] and Struble [2] who built on the early work of Clemmow [3] to develop generalised solutions for the equations as far as possible. In the second camp, the systems best known (and still used to check numerical models today) are those of Coppock [4], Corner [5] and Goldie [5], principally because they were described in the classic text book on internal ballistics by Corner [5]. The system of Taylor is described in “Interior Ballistics of Guns” [6], a handbook produced by the US Army in 1965. The system of Hunt-Hinds is described in the British equivalent, “Interior Ballistics” [7], produced in 1951. Excellent reviews of the various campaigns to develop analytical solutions, together with an extensive bibliography, is given by Bhaskara Rao and Sharma [8]

With the advances in fast, cheap computing power during the 1950s and ‘60s, it became easier to solve the ballistic equations numerically, and without having to sacrifice exactness by approximations or simplifications. However, it remains true that analytic models give an insight into the interrelations between the various parameters, and how they scale against each other, in a way which is not apparent in numerical systems. To this end, it is worth looking in more detail at the elegant system proposed by Mayer and Hart [9] which was used for many years in various guises at the Ballistics Research Laboratory in the USA and was considered very reliable in predicting accurate muzzle velocities. The derivation here differs slightly from that in the original paper in that it gives a somewhat more straightforward path through the mathematical thickets and brambles *en route* to the solutions.

Basic assumptions in the Mayer-Hart system

1) *The shot start pressure is zero.* That is, there is no separate variable for shot start pressure and it is assumed that the projectile starts to move immediately the powder starts to burn. This assumption is not really significant (in that it does not affect the maximum chamber pressure or muzzle velocity significantly) for large guns and cannons, or for the larger magnums in small-arms rifles, where the loading density is close to 100% and the working pressures are reasonable. It is of considerable significance for pistols, revolvers and the smaller rifle calibres (6mm and below) which use very fast powders, as the pressures can rise very quickly during the time it takes for the projectile to engrave the rifling of the barrel. This can be accounted for to some degree by an increase in the effective mass of the projectile, which has the effect of increasing the maximum pressure in the same way that a non-zero shot start pressure would, as will be described later.
2) *The covolume of the powder gasses is equal to the original charge volume.* This approximation is of no consequence when it matters, at the start of the bullet’s journey down the barrel, when the pressure is still rising quickly to peak pressure and relatively little of the powder has burnt. It only becomes significantly untrue when the bullet is well advanced down the barrel, at which time the system has effectively lost memory of the starting conditions anyway, so this is not a serious impediment.

3) *The burning rate of the powder is linearly proportional to the pressure.* This has been found to be a good approximation in practice for the pressures at which firearms generally operate, and is in any case a general assumption even in sophisticated numerical systems.
4) *The area of the burning surface remains constant.* This assumption is approximately true for "neutral" powder forms, such as flakes and cylindrical powders with one perforation, which are by far the most common powder kernel shapes in small-arms powders. Spherical ball powders have a naturally digressive shape, but this is mitigated by deterrent coatings which make such powders burn in more of a neutral or even slightly progressive manner.
5) *That no energy is lost through friction as the projectile travels up the barrel.* There is no separate variable for friction. This can be accounted for by increasing the effective mass of the projectile as will be described later.
6) *That no energy is lost through heat to the walls of the chamber or barrel.* There is no separate variable for heat loss. This can be accounted for by increasing the value of gamma, the ratio of specific heats, as will be described later.
7) *The projectile base pressure is the same as the chamber pressure.* This can be accounted for by an increase in the effective mass of the projectile, so that the projectile moves as if it were pushed by a lower pressure, as will be described later.

Units

Students of internal ballistics will be familiar with a bewildering array of pressure units used in the literature from bars to slugs through pascals, pounds per square inch, tons per square inch, kiloponds. . . . and more, raised to many different powers, with authors sometimes swapping casually between units in the course of a discussion. In the United States, it is common to use pounds per square inch (psi) for pressure and so it is convenient to use inches for the unit of length and pounds for the unit of weight. In those parts of the world where the metric system of units holds sway, Europe in particular, it is more usual to use the MKS system in the calculation, where distance is in metres and weight in kilograms. But the pressures most commonly used in non-technical ballistics literature is either psi or bar. One bar is just 14.5 psi and so easily related to psi at any stage in the calculation. Velocities in ft/sec. or metres/sec. are also easily related to the velocities of in/sec. used in the calculation. That being the case, it is actually most convenient to use units of inches and pounds in an internal ballistics system.

Glossary of symbols

A = the cross sectional area of the bore (in.^{2})
A_K = the surface area of a powder kernel (in.^{2})
C_0 = the total charge weight (lbs.)
d = bore diameter (ins.)
F = the Force of the powder (in-lbs/ lb.)
g = acceleration due to gravity (386.4 in/sec.^{2})
m_P = the mass of the projectile (lbs. One pound = 7000 grains)
$m_* =$ the effective mass accelerated up the barrel (lbs.)
P = gas pressure (lb/in.^{2})
T = temperature of the powder gasses (degrees Kelvin unless otherwise specified)
t = time (seconds)
v = velocity (inches/sec.)
v_{Muzzle} = muzzle velocity. (inches/sec.)
V = volume (in.^{3})
V_C = volume of case behind loaded projectile (in.^{3} The density of water is 252.8 grains/in.^{3})
V_K = volume of powder kernel (in.^{3})
x = projectile distance travelled up the barrel (ins.)
Z = fraction of powder charge burnt

α = exponent of pressure to determine powder burning rate
β = powder burning rate (ins/sec.)
γ = ratio of specific heats
ρ = density of powder (lbs/in.^{3} This is usually pretty close to 0.057 lbs/in.^{3}) Note: not bulk density
Λ = vivacity (fractional rate at which powder burns away per unit pressure per unit time)

The principal internal ballistics equations

Powder is burnt away in layers from the outside surface of the powder kernels. If $\overline{y}$ is a vector normal to the surface of a powder kernel, then the volume of the kernel being burnt away per unit time is $\beta A_K P$, where A_K is the burning surface area of the powder kernel, P is the gas pressure and β is a burning rate coefficient such that $\beta = \frac{d\overline{y}}{dt}$, the rate at which the kernel surface is ablated in the direction y, per unit of pressure.

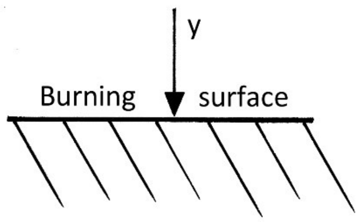
If we assume that each kernel of powder is exactly the same, and they are all ignited at the same time and burn at the same rate, then the rate at which the whole initial powder charge weight C_0 is burnt and converted into gasses will be equivalent to the rate at which the volume of each kernel is burnt way.

While the powder is burning, let C be the amount of powder charge left and let $Z = \frac{C_0 - C}{C_0}$.

When $C = C_0$ at the moment the powder starts burning, Z will be zero. When the powder is "all burnt" and C = 0, then Z = 1.

If Z is the fraction of powder charge that has been burnt away, then the rate at which the powder charge is being burnt will go as,

$$\frac{dZ}{dt} = \frac{\beta A_K P^{\alpha}}{V_K}$$ where V_K is the initial volume of the kernel.



Powder kernels burn from the outside in layers according to Piobert's law

There was much discussion by ballisticians at the end of the 19th century and the beginning of the 20th century, about what the value of α should be. A few general solutions of the ballistic equations have been attempted where α can assume any value. But by the middle of the 20th century it was generally agreed that the behaviour of smokeless powders in guns at the pressures of interest was well described by setting $\alpha = 1$, that is, the rate at which the powder burns is proportional to the chamber pressure, and that is the assumption in this system. In fact, modern measurements show that for the pressures of interest, α is quite close to one and usually has a value between 0.9 and 1.1.

It is assumed in this model that area of the kernel being burnt away remains constant with time, until the powder is "all burnt". This is approximately the case for flake powders where the thickness is small compared to the length or width. It is also approximately true for cylindrical powders with one perforation where the web thickness is small compared to the length. Such powder shapes are considered to be "neutral" in that they are not digressive, (like ball powders where the outer surface of the kernel decreases in area as it burns away), or progressive, (like cylindrical kernels with multiple perforations such that the burning area of the kernel actually increases with time).

For cylindrical kernels with one perforation, let the outer radius be r_0 and the radius of the perforation be r_i. Then assuming the length L of the kernel is large compared to the radius and burning at the ends of the kernel can be ignored, the burning area A_K of the kernel will be $2\pi L(r_0 + r_i)$, and the initial volume V_K will be $\pi L(r_0^2 - r_i^2)$

The rate of change for Z can now be written as,

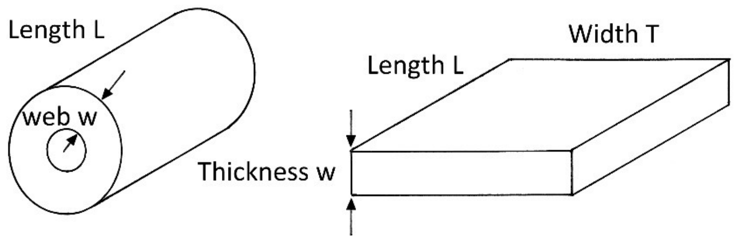
$$\frac{dZ}{dt} = \frac{2\beta P \pi L(r_0 + r_i)}{\pi L(r_0^2 - r_i^2)} = \frac{2\beta P}{(r_0 - r_i)} = \frac{2\beta P}{w}$$ where w is the web thickness of the kernel.

Similarly, for flake kernels with length L, width T and thickness w, where the thickness is small compared to the length and width, the rate of change of Z can be written as,

$$\frac{dZ}{dt} = \frac{2 L T \beta P}{L T w} = \frac{2\beta P}{w} \qquad (1)$$

The web thickness is an important parameter in powder kernel design as, for a powder having a given burning rate, this thickness defines the time for which the powder will burn, and so whether a powder is 'fast' or 'slow' can be determined by the web thickness.

In practice, for most small arms powders, the web or thickness w of the powder kernel is of the same order as the length L for flake or cylinder-with-one-perforation forms. The assumption in this system that $w \ll L$ is therefore not a good one. However, powders these days are usually coated with deterrents so that whatever their shape, they burn in a broadly neutral way and the assumption that powders are neutral in their burn rate is reasonably valid.



Two common "neutral" kernel shapes, cylindrical with one perforation and flake, where the burning area remains nominally constant. This assumes that the web/thickness w is much smaller than the other dimensions of the kernel.

Let V_0 be the initial free volume in the case behind the loaded projectile. Then;

$$V_0 = V_c - C_0 / \rho$$

Where V_c is the usable case capacity behind the loaded projectile, C_0 is the initial charge weight and ρ is the powder density. Once the powder has started burning and the projectile starts moving up the barrel, the volume behind the projectile will be V, where A is the area of the bore and x is the distance travelled up the barrel. (The covolume of the powder molecules as a gas is assumed to be the same as the volume taken up as a solid in the powder form.)

$$V = V_0 + A x$$

Rèsal's equation can be expressed as,

$$PV = C_0 Z F - (\gamma - 1)\left(\frac{m_* v^2}{2g}\right) \qquad (2)$$

where m_* is the effective mass of the projectile (see below) and F is the "Force" of the powder.

If the projectile did not move, then we could write $P_C V_0 = C_0 F$, where P_C would be the pressure reached inside what would be effectively a closed bomb of volume V_0

The equation of motion of the projectile will be,

$$\frac{m_*}{g}\frac{dv}{dt} = P A \qquad (3)$$

Equations (1), (2) and (3) are the principal internal ballistics equations from which the rest of the solution proceeds.

Solutions to the internal ballistics equations

The aim now is to get the pressure P in terms of the expansion ratio $\left(\frac{V}{V_0}\right)$ and we start by dividing equation (2) through by V_0, so that,

$$P\left(\frac{V}{V_0}\right) = \frac{C_0 Z F}{V_0} - \frac{(\gamma-1)}{V_0}\frac{m_* v^2}{2g}$$

$$= P_C Z\left[1 - \frac{(\gamma - 1)}{C_0 F Z}\frac{m_* v^2}{2g}\right] \qquad (4)$$

Now, $v = \frac{Ag}{m_*}\int P\,dt$ and $Z = \frac{2\beta}{w}\int P\,dt$

So let, $v = \frac{AgZw}{2\beta m_*}$ (Note that the velocity is proportional to Z) (5)

and equation (4) can be written,

$$P\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma - 1)}{2}\frac{A^2 g Z w^2}{4 C_0 F \beta^2 m_*}\right] \qquad (6)$$

If this is rewritten as,

$$\frac{P}{P_C}\left(\frac{V}{V_0}\right) = Z\left[1 - \frac{(\gamma - 1)}{2}\frac{A^2 g Z w^2}{4 C_0 F \beta^2 m_*}\right] \qquad (7)$$

it can be seen that equation (7) is dimensionless and thus we can define,

$$\frac{A^2 g\, w^2}{4 C_0 F \beta^2 m_*} = \left(\frac{P_C}{P_Q}\right)$$

where $P_Q = \left(\frac{2 C_0 F \beta}{A w}\right)^2 \frac{m_*}{g V_0}$ and is a constant with the dimensions of pressure.

Equation (7) can now be written as,

$$P\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right) Z\right] \qquad (8)$$

Now, from equation (3), the pressure can be expressed as,

$$P = \frac{m_*}{gA}\frac{dv}{dt}$$

and then with some deft manipulation using the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = Av\frac{dv}{d(Ax)} = Av\frac{dv}{dV} = Av\frac{dv}{dZ}\frac{dZ}{dV}$$

From equation (5)

$\frac{dv}{dZ} = \frac{Agw}{2\beta m_*}$, so we can write,

$$P = \left(\frac{Aw}{2\beta}\right)^2 \frac{Zg}{m_*}\frac{dZ}{dV} \;\rightarrow\; P_C\left(\frac{P_C}{P_Q}\right) Z V_0 \frac{dZ}{dV} \tag{9}$$

Substituting equation (9) into equation (8),

$$P_C\left(\frac{P_C}{P_Q}\right) Z V_0 \frac{dZ}{dV}\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right) Z\right]$$

Simplifying,

$$\frac{dZ}{dV} V = \left(\frac{P_Q}{P_C}\right) - \frac{(\gamma - 1)}{2} Z$$

Inverting and rearranging,

$$\frac{dV}{V} = \left[\frac{P_Q}{P_C} - \frac{(\gamma - 1)}{2} Z\right]^{-1} dZ$$

Now integrating,

$$\int_{V_0}^{V}\left[\frac{dV'}{V'}\right] = \int_0^{Z}\left[\frac{P_Q}{P_C} - \frac{(\gamma - 1)}{2} Z'\right]^{-1} dZ'$$

we get,

$$\ln\left(\frac{V}{V_0}\right) = -\frac{2}{(\gamma - 1)} \ln\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right) Z\right]$$

Taking anti-logs,

$$\left(\frac{V}{V_0}\right) = \left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right) Z\right]^{-\frac{2}{(\gamma-1)}} \tag{10}$$

We now have the expansion ratio as a function of Z. Rearranging so Z is a function of the expansion ratio,

$$Z = \frac{2}{(\gamma - 1)}\left(\frac{P_Q}{P_C}\right)\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right] \tag{11}$$

Pressure as a function of expansion ratio

We can now substitute for Z in equation (8) and achieve an equation for P in terms of the expansion ratio, so effectively get P in terms of x, the distance travelled by the projectile up the barrel.

$$P = \frac{2P_Q}{(\gamma - 1)}\left(\frac{V}{V_0}\right)^{-\gamma}\left[\left(\frac{V}{V_0}\right)^{\frac{(\gamma-1)}{2}} - 1\right] \qquad (12)$$

Energy as a function of expansion ratio

The energy of the projectile can be expressed from equation (5) and substituting for Z from equation (11)

$$\frac{m_* v^2}{2} = \frac{Z^2 P_C V_0 g}{2}\left(\frac{P_C}{P_Q}\right) \rightarrow \frac{2V_0 P_Q}{(\gamma - 1)^2}\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right]^2 \qquad (13)$$

Velocity as a function of expansion ratio

The velocity of the projectile as a function of the expansion ratio is then,

$$v = \left(\frac{4gV_0 P_Q}{m_*}\right)^{\frac{1}{2}} \frac{1}{(\gamma - 1)}\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right] \qquad (14)$$

Position of maximum pressure

Maximum pressure occurs when $dP / d\left(\frac{V}{V_0}\right) = 0$, so then

$$\left(\frac{V}{V_0}\right)_{P-Max} = \left[\frac{2\gamma}{(\gamma + 1)}\right]^{\frac{2}{(\gamma-1)}} \quad \text{and,} \quad Z_{P-Max} = \frac{P_Q}{\gamma P_C} \qquad (15)$$

Value of maximum pressure

The maximum pressure will have the value,

$$P_{P-Max} = P_Q\left[(\gamma + 1)^{(\gamma+1)} \gamma^{-2\gamma} 2^{-(\gamma+1)}\right]^{\frac{1}{(\gamma-1)}} \qquad (16)$$

This may be approximated to good accuracy (about one part in a thousand) by,

$$P_{P-Max} = \left(\frac{P_Q}{e}\right)\left[1 + \frac{3}{4}(\gamma - 1)\right]^{-1}$$

where $e = 2.718$ and is the base of the natural logarithm system.

Scaling factors that affect maximum pressure

Since $P_Q = \left(\frac{2 C_0 F \beta}{A w}\right)^2 \frac{m_*}{g V_C}\left(1 - \frac{C_0}{V_C \rho}\right)^{-1}$

it can be seen that the maximum pressure depends on the burning rate as β^2, on the web thickness as w^{-2}, on the Force as F^2 and on the inverse of the calibre to the forth power. It is proportional to the (effective) projectile weight, and depends on the charge weight as $C_0^2 / \left(1 - \frac{C_0}{\rho V_C}\right)$. The dependence on the chamber volume is as $\left(V_C - \frac{C_0}{\rho}\right)^{-1}$, which shows how the maximum chamber pressure rises rapidly with, and is very sensitive to the load when the load is near to filling the case.

Vivacity

These days, the closed bomb data which powder companies use to show the quickness of their powders is usually given in the form of vivacity charts, where the vivacity of the powder is simply the rate of change of Z plotted as a function of Z. The vivacity Λ is the fractional rate at which the powder volume burns away per unit pressure, per unit time. If we assume Λ is constant with Z, we can define,

$$\Lambda = \frac{2\beta}{w} \quad \text{and then} \quad \frac{dZ}{dt} = \Lambda P$$

In the expression for P_Q we can replace the burning rate β with, $\beta = \frac{\Lambda w}{2}$ so that, $P_Q = \left(\frac{C_0 F \Lambda}{A}\right)^2 \frac{m_*}{g V_0}$

Note that the web thickness w has dropped out of the equation. If vivacity is used instead of burning rate then there is no need for any knowledge of the powder kernel geometry at all. Note too that maximum pressure will scale as the square of the vivacity.

Deriving the burning rate from the maximum pressure

Mayer and Hart commented that if the powder vivacity or burning rate coefficient β is not known, then by rewriting P_Q in terms of P_{P-Max} in equation (16), it is possible to derive a value for β for a particular powder given a measured maximum pressure and other relatively well known parameters. (The powder Force and gamma do not change much from powder to powder, so may be assumed with relatively good accuracy). However, as commented below (see 'Accuracy'), the maximum chamber pressure is very dependent on whether the rate of burning when $Z \leq 0.2$ is progressive, digressive or neutral, and the burning rate obtained from the equations below will assume a neutral burning rate. Too, the burning rate and vivacity go as the square root of the maximum pressure and so are not very sensitive to it. At best, then, these expressions will give a starting point for the burning rate if the maximum pressure is known.

$$\beta = \frac{A w}{C_0 F}\left[\frac{0.68\, g V_0 P_{P-Max}}{m_*}\left(1 + \frac{3}{4}(\gamma - 1)\right)\right]^{\frac{1}{2}} \qquad (17)$$

Similarly, the vivacity Λ can be determined from,

$$\Lambda = \frac{2\,A}{C_0\,F}\left[\frac{0.68\,g\,V_0\,P_{P-Max}}{m_*}\left(1 + \frac{3}{4}(\gamma - 1)\right)\right]^{\frac{1}{2}}$$

Powder "all burnt"

The powder is "all burnt" when Z = 1 and this happens when the expansion ratio has reached a value of,

$$\left(\frac{V}{V_0}\right)_{All-Burnt} = \left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{-\frac{2}{(\gamma-1)}} \quad (18)$$

The pressure at "all burnt" is,

$$P_{All-Burnt} = P_C\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{\frac{(\gamma+1)}{(\gamma-1)}} \quad (19)$$

and the velocity at "all burnt" is,

$$v_{All-Burnt} = \left[\frac{g\,C_0\,F}{m_*}\left(\frac{P_C}{P_Q}\right)\right]^{\frac{1}{2}} \quad (20)$$

Beyond "all burnt", muzzle velocity and muzzle pressure

After all burnt, the amount of charge burnt does not increase beyond C_0 and so it must be true that Z stays at 1 thereafter. However, in equation (11) it can been seen that Z does continue to increase after all burnt. In consequence, these equations are not valid beyond all burnt. The usual course is to assume the volume expansion after all burnt is happening so quickly that the system can be treated as adiabatic (no heat loss to the walls) and so invoke the equation of adiabatic expansion $P_1\,V_1^{\gamma} = P_2\,V_2^{\gamma}$. This is an approximation that is made in all analytic models beyond all burnt.

Assuming that all burnt happens before the projectile reaches the muzzle, the equation of adiabatic expansion can be invoked to write muzzle pressure as,

$$P_{Muzzle} = P_C\left(\frac{V_0}{V_{Muzzle}}\right)^{\gamma}\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{-1} \quad (21)$$

and the muzzle velocity as,

$$v_{Muzzle} = \left(\frac{2\,g\,C_0\,F}{m_*(\gamma - 1)}\left\{1 - \left(\frac{V_0}{V_{Muzzle}}\right)^{(\gamma-1)}\left[1 - \frac{(\gamma - 1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{-1}\right\}\right)^{\frac{1}{2}} \quad (22)$$

These equations are of course applicable for any position past all burnt, as well as the muzzle.

The resulting plot of pressure as a function of distance travelled shows a discontinuity in the rate of change of pressure at the position of all burnt. This discontinuity has never been observed in practice, but then the system is not adiabatic at this stage (there <u>are</u> heat losses to the walls) or at any stage for that matter. Too, the powder kernels in practice are not all of exactly the same shape, burning at exactly the same rate, and reaching all burnt at exactly the same time as this system supposes, so there is no single point at which all burnt may be said to occur.

Scaling factors that affect muzzle velocity for a given calibre

It can be seen that for long barrels, where $V_{Muzzle} \gg V_0$, the muzzle energy (which goes as v^2) is proportional to the charge weight, and to the Force of the powder. However, the muzzle energy will be independent of the (effective) projectile weight for any given charge weight.

In reality, the barrel friction will reduce the muzzle energy of the projectile by a fixed amount, regardless of charge weight. The muzzle energy will still be a linear function of charge weight, however, even if it is not actually proportional to charge weight.

How parameters scale with calibre – "Ballistic similitudes"

When designing a new large gun, considerable savings in the costs of research and development are possible by building test guns on a much smaller scale. Suppose a shell of a given weight was required to be launched from a gun of a certain calibre and length at a given muzzle velocity. How could this be tested using a gun scaled down in dimensions by a given factor so that the maximum chamber pressure and muzzle velocity would be the same in the full size gun as they are found to be in the miniature test gun?

The full size gun and the miniature test gun would have the same expansion ratios. The barrel length would be reduced in proportion to the calibre, and the chamber volume would be reduced in proportion to the cube of the calibre. It is assumed that the same powder composition would be used, so the burning rate β and the powder Force F would remain the same.

The peak pressure is proportional to P_Q which goes as, $P_Q = \left(\frac{2C_0 F\beta}{Aw}\right)^2 \frac{m_*}{gV_0}$ from which we can see that the effective mass (and so projectile mass) must be proportional to the initial volume V_0 and so must scale as the cube of the calibre. From equation (22) for the muzzle velocity, it can be seen that the ratio of charge weight to effective mass must remain constant, so the charge mass must then also scale as the cube of the calibre. We also see in the equation for P_Q that the web thickness w must scale as the calibre, since the product of w and the bore area A must scale in proportion to the charge weight. Finally, since the web thickness must scale as the barrel time for a propellant of a given burning rate, we can deduce that the time to any given expansion ratio must scale as the calibre.

For example, suppose we wish to find what powder charge, burning rate and web thickness is required for a 6" calibre naval gun firing a projectile of 100 lb weight for a specified velocity of 2740 ft/sec from a barrel of length 50 calibres (300 inches or 25 feet)? The initial chamber volume behind the loaded shell is about 1000 in^3.

Suppose we build a test gun on a scale 1:20, which would then have a .308" calibre. Barrel length will be 15 inches. The usable case capacity will need to be one eighth of a cubic inch or 31.5 grains of water, (which happens to be the usable case capacity of the 7.62 x 39 cartridge). After some experimentation, we find that 26.3 grains of a powder with a web thickness 0.01" and a burning rate of about 4.2 x 10^{-4} in./sec./psi. will produce a peak pressure of about 40,000 psi. when used behind a scaled down projectile weighing an 80th of a pound, or 87.5 grains, and will produce a muzzle velocity of 2740 ft/sec. Such a powder would have a vivacity of about 123 per 100 bar/sec.

Scaling up then, we will need a charge weight of 30 lbs of powder having a kernel web thickness of about 0.2" for the same powder composition in the 6" naval gun, which will then have the same peak pressure of 40,000 psi and the same muzzle velocity of 2740 ft/sec. In fact, the powders used in large guns of this kind have burning rates of the order of 1.5 x 10^{-4} in./sec./psi., so the web thickness could be reduced accordingly to about 0.06". If the measured barrel time for the miniature test gun was one millisecond, we could expect the barrel time for the parent 6" naval gun to be about 20 milliseconds. These are in fact quite typical parameters for this type of gun.

This principle was used by the Krupp factory in Germany, when in the mid 1930s they were called upon to develop (amongst other big guns) an 80cm calibre railway gun to smash the defences of the French Maginot line, then the heaviest fortifications in existence, in the anticipated Battle of France. The gun had a barrel length of 32.5 metres and fired a shell weighing 7 tonnes with a muzzle velocity of 2700 ft/sec. It was finished too late for the Battle of France, but was used with great success in other campaigns throughout the Second World War. To verify the design of the gun, Krupp built a 1/10th scale test gun with a calibre of 8cm. Krupp later claimed that the propellant and charge so deduced was correct for the full scale gun without the need for any further alteration [7].

Pressure as a function of time

Pressure as a function of time has not been discussed so far. Mayer and Hart commented that if the pressure as a function of distance travelled was known, together with the velocity as a function of distance travelled, then determining a pressure-time relationship was "always possible". In numerical solutions of the basic ballistic equations, time is the fundamental variable. But a solution for time is not given in most analytic systems. This might seem odd as a solution for time would seem to be relatively straightforward.

Starting (for example) from the basis that,

$$v = \frac{dx}{dt} = \frac{dx}{d(V/V_0)}\frac{d(V/V_0)}{dt} \qquad (23)$$

then using $V = V_0 + Ax$ and dividing through by V_0,

$$\frac{V}{V_0} = 1 + \frac{A}{V_0}x$$

and differentiating, $\dfrac{d(V/V_0)}{dx} = \dfrac{A}{V_0}$ (24)

Substituting equation (24) into equation (23), and rearranging gives,

$$\int_0^t dt' = \frac{V_0}{A}\int_{V'=V_0}^{V'=V} \frac{d(V'/V_0)}{v}$$

Now, substituting for v from equation (14), the integration can proceed, yielding an expression which gives time as a function of the expansion ratio, from which pressure as a function of time may be readily determined.

$$t = \left[\frac{2V_0}{A v_k}\left(\frac{V'}{V_0}\right)^{\frac{(\gamma+1)}{2}} \ln\left(1 - \left(\frac{V'}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right)\right]_{V_0}^{V}$$

where $v_K = \left(\dfrac{4 g V_0 P_Q}{m_*}\right)^{\frac{1}{2}}$

It is immediately apparent that there is a singularity at $V' = V_0$ and the resultant equation for time as a function of V/V_0 is not well behaved.

Since all the analytic solutions of the ballistic equations are necessarily of the same general form as those given above, this is a common problem with all analytic systems and not just a problem with the Mayer-Hart system. Struble [2] proved what Mayer and Hart probably suspected, that if it is assumed that the burning rate of the powder is proportional to the pressure, it is actually not possible to derive a closed analytic form for time. In some systems a function for time is fudged. Corner, for example, avoided the singularity by truncating his solutions so that both v and P fail to vanish at t = 0.

The best way to proceed in a practical sense is to start at the muzzle and simply calculate the values of 1/v for the half-inch values of x between the integer values. These values will then be (very nearly) equivalent to the times taken for the projectile to travel the distance between the integer values of x. By also calculating pressures at integer values of x, the pressure as a function of time can then easily be plotted from the muzzle backwards. As peak pressure it usually reached very soon after the projectile has started moving, it may be necessary to use this general principle, but for smaller steps of distance, where the pressure is rising quickly to the maximum.

Accounting for shot start pressure, barrel friction and pressure gradients behind the projectile.

To account for the fact that the projectile base pressure, or shot pressure, will be less than the mean pressure in the barrel (see below), the mass of the projectile can be increased, so that it accelerates as though it were being pushed by a lower pressure. It is also common to increase the projectile mass to account for the energy expended in overcoming friction in the barrel, and the energy expended in recoil. (This assumes that the friction is proportional to the force on the projectile base. Experimental evidence would appear to show that this is actually a reasonable assumption as friction seems to decrease with velocity, which is also true for force on the projectile base.)

To this end it is usual to increase the projectile mass by 5%

Accounting for gas kinetic energy

It can be shown that the kinetic energy of the gasses and powder at any time is such that a third of the powder charge may be considered to be travelling down the barrel at the same velocity as the projectile. The effective mass m_* can then be written as,

$$m_* = 1.05 m_P + \frac{1}{3} C_0$$

Where m_P is the projectile weight.

Accounting for the difference between mean gas pressure and chamber (breech) pressure

The model assumes that the gas pressure is uniform along the whole column of gas from the breech to the base of the projectile. In fact, the gas at the breech end will be stationary (no net kinetic translational energy) whereas the gas immediately behind the projectile is necessarily moving at the projectile velocity. It follows that there will be a gradient of pressure decreasing from the breech to the projectile base.

It can be considered that the pressures calculated above are actually the mean pressure P_M of the gasses behind the projectile, as defined by Lagrange, such that the chamber pressure P_B is related to the mean pressure by,

$$P_B = P_M \left[\frac{1 + \dfrac{C_0}{2m_P}}{1 + \dfrac{C_0}{3m_P}} \right]$$

Using this expression, the calculated maximum mean pressure in equations (16) can be corrected to give the breech or chamber pressure that would be measured using a piezo pressure gauge.

Accounting for heat loss to the barrel walls

If heat losses Eh are included in Rèsal's equation, it would be expressed as,

$$PV = C_0 F Z - \frac{(\gamma - 1)}{g}\left[\left(\frac{m_* v^2}{2}\right) + Eh\right]$$

Kent and Vinti [10] proposed using a trick that has long been used to account for heat loss in gas engines, whereby an increase in gamma would effectively mimic an energy loss in the system. They assumed that if the heat loss in the barrel is proportional to projectile energy, then Rèsal's equation can be written,

$$PV = C_0 F Z - \frac{(\gamma - 1)}{g}\left[(1 + k)\left(\frac{m_* v^2}{2}\right)\right]$$

Where k is the ratio of the heat loss Eh to the kinetic energy of the effective mass. A new effective gamma γ_{EFF} can be defined where

$$(\gamma_{EFF} - 1) = (\gamma - 1)(1 + k) \qquad (25)$$

And now Rèsal's equation can be written in its original form as per equation (2), but with γ_{EFF} instead of γ .

There are two common routes to find a value for k. One is to recall that the ratio of heat loss to muzzle energy in small arms is around 40% of the projectile energy. (In large guns, it is around 10%). The value of k can then be set accordingly and a value for γ_{EFF} derived from equation (25) for use in the system.

The other is to invoke the Thornhill equation for heat loss to the barrel, which is a semi-empirical function where, conveniently, the heat loss is given as being a function of v^2 and so proportional to projectile energy. This equation is taken from Corner [5].

$$Eh = \frac{3.034\, d^{3/2}\left(x - \frac{V_C}{A}\right)(T - T_0)}{1.7 + 0.38\, d^{1/2}\left(\frac{d^2}{Z C_0}\right)^{0.86}} \frac{v^2}{v^2_{Muzzle}} \qquad (26)$$

T is the gas temperature in degrees Centigrade, usually set to about 2700 °C, and T_0 is the initial temperature of the barrel. The energy Eh here is in in-lbs.

The value of k used to determine γ_{EFF} will depend on the muzzle energy and so on the muzzle velocity, but the muzzle velocity from equation (22) depends on γ_{EFF} . So, a process of iteration is required where trial muzzle velocities are used to determine k until there is agreement between the trial muzzle velocity and the muzzle velocity found using the resultant γ_{EFF} from equation (22). For small arms, γ_{EFF} should have a value of about 1.35.

Accuracy

With the adjustments described above to account for barrel friction, heat loss etc., the system should give muzzle velocities that are accurate to 2 – 3%. It is actually quite difficult to do much better than this even with "exact" numerical models, due to the uncertainties in determining burning rates, ratios of specific heats, and other constants – most of which are not actually constant as the powder burns and the pressure varies.

Peak pressure is very sensitive to the extent to which powders are neutral in their burning, or lean to being progressive or digressive, during the rapid rise in pressure before peak pressure is reached, and which happens when the fraction of powder burnt is between 20% and 30%. Progressive powders, where vivacity is increasing with

Z, will have a slower rise to peak pressure and a consequently lower peak pressure. Digressive powders have a rapid rise to peak pressure and much higher peak pressures thereby.

However, matters are complicated as closed bomb results very early and very late in the powder burning history are not considered reliable, and the calculated vivacity for at least the first 10% and perhaps the first 20% of the powder burn $(Z \leq 0.2)$ is usually discounted. Even with detailed numerical models it is then difficult to accurately predict the peak pressure with poor knowledge of the powder vivacity during this crucial time in the burning history. For this system, where the burning rate is assumed to be neutral (constant with Z), the predicted peak pressures should be regarded as "indicative" and perhaps accurate to $\pm 30\%$.

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