A Numerical System for the Internal Ballistics of Small Arms

Introduction

While much effort was spent on analytic solutions to the internal ballistics equations during the first half of the 20th century, a solution was never possible in a closed form without some constraints, approximations or simplifications. With the development of digital computers after WWII, "exact" solutions of the ballistics equations without approximations became possible using numerical methods, with realistic expectations of time and effort involved. Analytic solutions are still used today as a quick check to see that the numerical solutions are within the realms of expectation, but the development of analytic solutions effectively ended with the advent of reasonably cheap, quick, computer power. Analytic solutions also have the advantage of being precisely correct, within the limits of their boundary conditions, so can be used to check the accuracy of a numerical method subject to the same boundary conditions.

The first campaign to achieve a numerical solution on a digital computer was probably that reported by Baer and Frankle [1] on the ORDVAC computer, which had been built by the University of Illinois for the US Ballistics Research Laboratory in 1952 - though was much updated by the time Baer and Frankle wrote their report ten years later. This program is still in use today, though in modified form [2] and is known as the "IMHVG2 Lumped Parameter Model". However, the computation routine outlined then has essentially been followed in almost every lumped parameter internal ballistics computer model ever since.

The basis of the computational routine is that differential equations in time for projectile position, velocity, powder burn rate, volume behind the projectile, chamber pressure and bullet base pressure are solved simultaneously, usually using a standard Runge Kutta matrix ordinary differential equation solver package which is available these days for most modern computer languages.

Systems such as this assume either a uniform powder gas density, or a fixed pressure gradient. This is an adequate description for most small arms, where essentially we are only interested in the maximum chamber pressure, the muzzle velocity, and perhaps the position of projectile travel at maximum pressure. This is the essence of what Corner called the "classical ballistics" problem [3]. Much effort in computational ballistics since WWII has been spent in trying to understand the pressure waves which can propagate in the gas column behind the projectile, which can build up with positive feedback to catastrophic levels. This is particularly a problem in big guns. The observation of such pressure waves was reported by Petavel in 1905 [4] and were described theoretically in a seminal paper by Love and Pidduck in 1923 [5]. This paper formed the basis for much of the effort to understand this phenomena in modern hydrodynamic computer systems using finite element techniques.

The solution presented here is a lumped parameter system, but would appear to be novel in that it can be shown that only one differential equation needs to be solved, instead of a number simultaneously, and that can be done using a simple quadratic interpolation which just needs a few lines of code. This means that accurate and 'exact' solutions to the internal ballistics equations can easily be performed on small computers or even programmable calculators.

After 'all burnt' it is usual in lumped parameter systems to assume that the volume expansion is happening so quickly that the system can be treated as adiabatic (no heat loss to the walls). This simplifies the ballistic equations greatly since the equation of adiabatic expansion, $P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$, can be invoked. This approximation was useful when analytic solutions has to be laboriously calculated by hand, but it has now become set in stone as the accepted formal approach so that it continues to be used today - even though the expansion is not adiabatic (there <u>are</u> heat losses to the barrel wall), and the approximation makes effectively no difference at all to the computation time when running a numerical model on a modern computer. Accordingly, this system does not assume the expansion is adiabatic at any stage.

Units

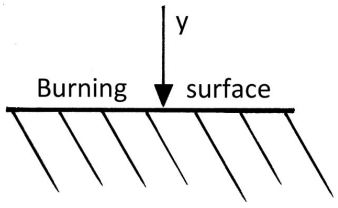
Students of internal ballistics will be familiar with a bewildering array of pressure units used in the literature from bars to slugs through pascals, pounds per square inch, tons per square inch, kiloponds. . . . and more, raised to many different powers, with authors sometimes swapping casually between units in the course of a discussion. In the United States, it is common to use pounds per square inch (psi) for pressure and so it is convenient to use inches for the unit of length and pounds for the unit of weight. In those parts of the world where the metric system of units holds sway, Europe in particular, it is more usual to use the MKS system in the calculation, where distance is in metres and weight in kilograms. But the pressures most commonly used in non-technical ballistics literature is either psi or bar. One bar is just 14.5 psi and so easily related to psi at any stage in the calculation. Velocities in ft/sec. or metres/sec. are also easily related to the velocities of in/sec. used in the calculation. That being the case, it is actually most convenient to use units of inches and pounds in an internal ballistics system.

Glossary of symbols

A = the cross sectional area of the bore (in.^{2})
A_K = the surface area of a powder kernel (in.^{2})
b = the covolume of the powder gasses (in^3 / lb.)
C_0 = the total charge weight (lbs.)
F = the Force of the powder (in-lbs/ lb.)
g = acceleration due to gravity (386.4 ins/ sec/ sec.)
m_P = the mass of the projectile (lbs.)
$m_* =$ the effective mass accelerated up the barrel (lbs.)
m_W = molecular weight for nitrocellulose powder gasses (22.7 for the gasses of nitrocellulose powder)
P = gas pressure (lb/in.^{2})
R = gas constant (33377 in-lbs. force/ lb/ ^{0}C)
T = temperature of the powder gasses (degrees Kelvin)
t = time (seconds)
v = velocity (inches/sec.)
V = volume (in.^{3})
V_C = volume of case behind loaded projectile
V_K = volume of powder kernel (in.^{3})
x = projectile distance travelled up the barrel (ins.)
Z = fraction of powder charge burnt
α = exponent of pressure to determine powder burning rate
β = powder burning rate (ins/sec.)
γ = ratio of specific heats
ρ = density of powder (lbs/in.^{3} This is usually pretty close to 0.057 lbs/in.^{3}) Note: not bulk density
Λ = vivacity (fractional rate at which powder burns away per unit pressure per unit time)

The burning of smokeless powders

Smokeless powders burn by deflagration, such that the outside of the powder kernel burns away, reducing the volume of the kernel until there is nothing left. If $\bar{y}$ is a vector perpendicular to the surface of the powder kernel, and $\frac{d\bar{y}}{dt}$ is the rate at which the powder kernel burns away perpendicular to the surface of the kernel,



Powder kernels burn from the outside in layers according to Piobert's law

let $\frac{d\bar{y}}{dt} = \beta P^{\alpha}$ where β is the burning rate of the powder per unit of pressure P. This is known as Vieilles Law.

The value of α was a matter of hot dispute amongst ballisticians for much of the first half of the 20th century, but it gradually emerged that setting $\alpha = 1$ in ballistics models gave good comparisons with experiments and later measurements with modern equipment have shown that α is usually between 0.9 and 1.0 for smokeless powders in small arms for the pressure ranges of interest. For the discussion about this system, we will follow the usual form of setting $\alpha = 1$, though it is possible to set alpha to any value without further complication.

The rate at which the volume of the original powder charge C_0 is burnt away with then go as $d\bar{y}/dt$ times the total area of the powder charge. But if we assume that all the kernels of powder are the same, and that they all start burning at the same time and burn away at the same rate, then the rate at which one powder kernel burns away is the same as the rate at which the powder charge as a whole burns away. It is thus only necessary to consider the burning history of one kernel of powder.

If $A_K(t)$ is the surface area of the kernel at time t, then the rate at which the volume of the kernel is burning away at any time t is $\beta\ P(t)\ A_K(t)$.

Let C(t) be the amount of powder charge left after time t, and let $Z(t) = \frac{C_0 - C(t)}{C_0}$.

Z will be zero at t = 0 and when $C(t) = C_0$. When the powder is "all burnt" and C(t) = 0, then Z = 1.

The rate at which the powder burns away is then,

$$\frac{dZ}{dt} = \frac{\beta\, A_K(t)\, P(t)}{V_K}$$

These days, powder companies publish the burning rates of their powder in terms of the relative vivacity of the powder. The vivacity is the fractional rate at which the powder reduces in volume per unit pressure. The relative vivacity relates this rate to the original volume. It is graphed as a function of Z and the burning rate β at any stage in the burning process is related to the vivacity by,

$$\beta(Z) = \text{Vivacity}(Z)\ \frac{V_K}{A_K}$$

where V_K and A_K are the original volume and area of the kernel before burning starts.

The internal ballistics system

The system described here calculates the internal ballistics dynamics in program iterations or loops during successive short periods Δt. The loop detailed here is from a time t to a time $t + \Delta t$.

The burning rate $\beta(Z)$ is determined for the value of Z at time t where,

$$\beta(Z) = \text{Vivacity}(Z)\ \frac{V_K}{A_K} \qquad (1)$$

The relative vivacity as a function of Z is given by the powder companies and is stored as a function or as a table for use by the program. The change of Z will be very small over the period Δt, so little accuracy is lost by assuming the relative vivacity, and so the burning rate, is constant over the period Δt.

The dimensions of the powder kernel will generically be w_1, w_2 and w_3 (for example, length, width and thickness for flake powders) and these dimensions will reduce during period Δt as the kernel burns away such that,

$$w_{1,2,3}(t+\Delta t) = w_{1,2,3}(t) - 2\,\beta(Z)\int_{t}^{t+\Delta t} P(t')\,dt \qquad (2)$$

The volume of the kernel at the end of the period Δt will be,

$$V_K(t+\Delta t) = \text{function}\left(w_1(t+\Delta t),\ w_2(t+\Delta t),\ w_3(t+\Delta t)\right)$$

The value of Z at the end of the period Δt will be

$$Z(t+\Delta t) = \left[1 - \frac{V_K(t+\Delta t)}{V_K(t=0)}\right] \qquad (3)$$

This value of Z is used to determine the vivacity to be used for the next period Δt. The projectile will be accelerated to a new velocity due to the pressure P_B on its base, where

$$v(t+\Delta t) = v(t) + m_P\,A\,\Psi\int_{t}^{t+\Delta t} P(t')\,dt \qquad (4)$$

Ψ is a function which relates P_B the projectile base pressure to the chamber pressure P.

The distance the projectile has travelled at the end of the period Δt is described with good accuracy by,

$$x(t+\Delta t) = x(t) + \left[\frac{v(t) + v(t+\Delta t)}{2}\right]\Delta t \qquad (5)$$

The temperature of the gasses can be determined from

$$T = \frac{P\,g(V-b)m_w}{R}$$

Where V, the volume behind the projectile, can be substituted to produce,

$$T(t+\Delta t) = \frac{P(t+\Delta t)\,g\,m_w\left[V_C + Ax(t+\Delta t) - \frac{C_0}{\rho} - C_0\,Z(t+\Delta t)\left(b - \frac{1}{\rho}\right)\right]}{R\,C_0\,Z(t+\Delta t)} \qquad (6)$$

The value of $P(t+\Delta t)$ in equation (6) is the interpolated value of pressure, derived from equation (8) below.

Using equations (3), (4) and (5) for the values of Z, the projectile velocity and the distance travelled, the actual chamber pressure $P(t+\Delta t)$ at the end of the period Δt can now be determined from Rèsal's equation.

$$P(t+\Delta t) = \frac{F\,C_0\,Z(t+\Delta t) - (\gamma-1)\left[\frac{1}{2}m_P v(t+\Delta t)^2 + \Phi\,x(t+\Delta t) + Eh(t+\Delta t)\right]}{g\left[V_C + Ax(t+\Delta t) - \frac{C_0}{\rho} - C_0\,Z(t+\Delta t)\left(b - \frac{1}{\rho}\right)\right]} \qquad (7)$$

The term Φ is the frictional force between the projectile and the barrel and is considered not dependent on projectile velocity.

The term Eh is the heat lost to the barrel walls from the propellant gasses, per unit time. The Thornhill equation [3] is commonly used, which assumes heat loss to the barrel is a function of projectile energy. More meaningful functions are dependent on the gas temperature though, which is obtained from equation (6).

In this way the progress of the projectile up the barrel is plotted through successive loops or iterations in the computer program.

It will be seen that in all the ballistics equations given above, the only differential equation that needs solving (integrating) is $\int_{t}^{t+\Delta t} P(t')dt$. All the other variables are dependent only on this one differential equation. But the pressure has only been calculated up to the time t before the start of this particular loop and to solve this integral, pressures from time t to time $t + \Delta t$ are needed.

The solution is approximated by a Runge Kutta interpolation, using previously stored values of pressure from times $(t-2\Delta t)$, $(t-\Delta t)$ and t, as determined using equation (7) in previous loops. A quadratic equation of the form $P = at^2 + bt + c$ is fitted to these stored values. Pressures at times later than P(t) can be interpolated such that,

$$P(t + \frac{\Delta t}{2}) = 2.25P(t-2\Delta t) + 1.875\left[P(t) - P(t-2\Delta t)\right] - 1.25\,P(t-\Delta t) \qquad (8)$$

and

$$P(t+\Delta t) = 4P(t-2\Delta t) + 3\left[P(t) - P(t-2\Delta t)\right] - 3P(t-\Delta t) \qquad (9)$$

Simpsons Rule can then be used to calculate the integral such that

$$\int_{t}^{t+\Delta t} P(t')dt = \frac{\Delta t}{6}\left[P(t) + 4P(t+\frac{\Delta t}{2}) + P(t+\Delta t)\right] \qquad (10)$$

If Δt is set sufficiently small, then a further simplification can be made where we apply the trapezium rule,

$$\int_{t}^{t+\Delta t} P(t')dt \simeq P(t)\Delta t \qquad (11)$$

The speed of modern computers is such that the run-time is not an issue if $\Delta t = 10^{-7}$ seconds, or even 10^{-8} seconds, which means that for a small arms system there might be around ten or even a hundred thousand iterations or loops. However, if Δt is made arbitrarily small, errors through the limits of number precision start to become apparent and this is the fundamental limit on accuracy using the trapezium rule.

Benchmark validation

This numerical system can be tested by comparing it with a precise analytic solution of the ballistics equations, which can be closed subject to certain boundary conditions. For this purpose, the analytical solutions by Mayer and Hart [6] are used.

The boundary conditions are that the shot start pressure is zero, that the covolume of the gasses is equal to the original charge volume, that the burning rate of the powder is proportional to the pressure, that the area of the burning surface of the powder remains constant (“neutral” burning), and that no energy is lost to the chamber or barrel through heat loss. Adiabatic expansion is assumed after all burnt in the Mayer and Hart analytic system, but not in the numerical system.

To account for the kinetic energy of the powder and powder gasses, which are also being accelerated up the barrel, the projectile mass is replaced by an “effective mass” which includes one third of the charge mass. A common approximation to account for barrel friction and projectile base pressure is to assume it is proportional to the force on the base of the projectile, and add 5% to the projectile mass. The effective mass m_* is then,

$$m_* = 1.05m_P + \frac{1}{3}C_0$$

It is also assumed that the pressure calculated by both the numerical and the analytic systems is actually the mean pressure P_M of the gasses, as defined by Lagrange, such that the breech pressure P_B is related to the mean pressure by,

$$P_B = P_M \left[\frac{1 + \frac{C_0}{2m_P}}{1 + \frac{C_0}{3m_P}} \right]$$

For this test, both systems modelled a typical 308 Winchester cartridge, with calibre .308 inches, having usable case capacity of 51 grains of water. Projectile weight was 150 grains. Barrel length was 30 inches. The powder charge was 46 grains. The powder had a vivacity of 71 per 100 bar/sec. The powder Force was 3,367,500 in-lbs/lb., gamma was 1.24 and powder density 0.0584 $lb/in.^3$ The results were as follows with various values of Δt for the numerical system, on a 64 bit computer using Perl with double precision.

	Mayer-Hart	10^{-6} sec.	$5x10^{-6}$ sec.	10^{-5} sec.	$5x10^{-5}$ sec.
Breech pressure P-Max	48,196 psi	48,208	48,248	48,275	48,006
Muzzle velocity	2899 ft/sec.	2898	2900	2902	2903
Travel to P-Max	1.5 inches	1.5	1.5	1.5	1.5
Velocity at P-Max	925 ft/sec.	962	962	976	949
Muzzle pressure	5820 psi	5834	5816	5787	5710
Powder all burnt at	6.5 inches	6.5	6.5	6.5	5.9

It can be seen that with a one microsecond time step, agreement is almost perfect with the precise analytical solution. Even with time steps increased to 50 microseconds, about one twentieth the time it takes for the projectile to travel the length of the barrel, maximum breech pressure is still correct to about 4 parts per 1000, and muzzle velocity is correct to better than 2 parts per 1000. Time steps longer than 50 microseconds start to become comparable to the pressure rise-time however, so accuracy then drops off very quickly.

References

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