SIMPLIFIED EQUATIONS OF INTERIOR BALLISTICS.

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ABSTRACT.

Subject to the assumptions of zero starting pressure, covolume equal to charge volume, burning rate proportional to pressure, constant burning surface, and no heat loss through gas or projectile friction, the equations of interior ballistics are presented in simple form.

It is hoped that these simple equations will be of some aid in making ready approximate evaluations of the effect of variation of any parameter.

The general equations are listed in section I. In section II the expressions for the maximum pressure and muzzle velocity are discussed more fully, and the derivatives with respect to various parameters are given.

I. THE GENERAL EQUATIONS.

The equations of interior ballistics are used to calculate the velocity of the projectile as a function of travel in the bore of the gun tube, and the pressure in the tube as a function of the travel.† The conversion to pressure as a function of time is important for some problems, such as recoil, and is always possible if the two functions mentioned above are available. In this paper equations are given for the velocity-travel and pressure-travel curves, but not for the pressure-time relationship.

The projectile is propelled by the pressure of gas acting on its base. The pressure of gas is affected by the amount of charge burnt, which in turn is a function of the past pressure-time relation, since the rate of burning of propellants depends on the pressure. The equation of state of the gas, $p(v - cb) = cRT$, is commonly used (with p = pressure, v = volume, b = a temperature-independent covolume of the gas, c = charge, R = a constant, and T = absolute temperature). The temperature of the gas is T_0 if no work has been done and no heat lost. This temperature is referred to as the adiabatic flame temperature. The somewhat inconsistent term specific force, or just force, λ, is used for the product RT_0. The specific heat of the gas is implicitly included by giving the ratio of specific heats γ, so that the decrease in pv due to the loss of energy of the gas is equal to $(\gamma - 1)$ times the energy loss (equation (5)).

The propellent charge of a gun is in the form of powder grains of known size and shape. Each of these grains burns on its surface,

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† T. J. Hayes, "Elements of Ordnance," New York, John Wiley & Sons, Inc., 1938. C. Cranz, "Lehrbuch der Ballistik," Vol. II, Berlin, J. Springer, 1926. The latter work contains an extensive bibliography.

which recedes parallel to itself at a rate dependent on the gas pressure. Thus the actual development of gas depends on the geometry of the grains, and grains are used for which the surface decreases (regressive grains) during burning, stays constant (constant burning grains) or even increases (progressive grains). The treatment here is for grains of constant burning surface. The grains are usually described by giving their "web," w, which is essentially their thickness, that is, twice the amount of surface regression necessary before two opposite burning surfaces come into contact.

The loss of efficiency through heat conduction and friction is not large in artillery weapons, and is neglected here. In actual rifled weapons the projectile starts to move only after the pressure exceeds a certain "starting pressure," an effect which is also neglected in what follows.

The differential equations, which are set up in accord with the commonly used physical hypotheses, are usually solved by numerical integration, and the results tabulated for various values of the many parameters involved. By a suitable choice of assumptions relatively simple analytical solutions may be obtained. These are convenient, even if highly approximate.

We use the symbols

a = cross section of the bore (ins.)^2.
b, as a subscript, indicates all burnt.
c = charge burnt (lbs.).
C = total charge (lbs.).
g = gravitational constant = 386.4 in./sec.^{2}
m, as a subscript, refers to muzzle.
max, as a subscript, refers to maximum pressure.
p = pressure (lbs./in.^{2}).
p_c = a constant of the dimensions of pressure (lbs./in.^{2}).
p_q = a constant of the dimensions of pressure (lbs./in.^{2}).
q = burning constant (surface recession (ins./sec.) = $q \times$ pressure (lbs./in.^{2})).
T = temperature, absolute, of the powder gases.
T_0 = the adiabatic flame temperature.
u = velocity of the projectile (ft./sec.).
v = volume behind projectile base minus charge volume (ins.)^3.
v_c = chamber volume (ins.)^3.
v_0 = chamber volume minus original charge volume (ins.)^3.
W = projectile weight plus one third C (lbs.).
w = web (ins.).
γ = ratio of specific heats.
λ = specific force of propellant (ins.); this is defined as the pressure volume product of unit weight of propellant if burned without loss of energy.

We assume:

(1) that the starting pressure is zero; that is, that the projectile starts to move immediately the powder begins to burn and that the pressure necessary to engrave the band is zero,

(2) that the covolume of the propellent gases is equal to the original charge volume,

(3) that the chemical burning rate of the propellant is proportional to the first power of the pressure,

(4) that the area of the surface of the propellant remains constant during the burning,

(5) that no heat or energy is lost through conduction or through friction.*

The burning of a propellant is given by the statement that the rate of recession $\dot{z}$ of the burning surface of the grain is a function of the pressure, and we choose

$$\dot{z} = qp, \tag{1}$$

with q the burning constant of the powder. The amount, $\dot{c}$, of charge burnt per unit time is $\dot{z}\rho\sigma$, where ρ is the propellant density and σ its surface. Since the propellant is assumed to be of such a shape that its surface remains constant (for instance, sheets of area $\gg$ than thickness squared) the volume of the propellant is initially $(w\sigma/2)$ with w the web or thickness, and the initial charge C is $(\rho w\sigma/2)$. One then has $\rho\sigma = 2C/w$ and

$$\dot{c} = C(2\dot{z}/w) = C(2qp/w) \tag{2}$$

from (1).

The specific force λ of the propellant is the product of p and the free volume for the uncooled gases produced from unit weight of propellant. The free volume of the chamber, v_0, is

$$v_0 = v_c - 17.7C \tag{3}$$

where 17.7 is the average volume in cubic inches of one pound of propellant, and the free volume behind the projectile base, v, is

$$v = v_0 + 12as \tag{4}$$

where s is the travel in feet.

* The common procedure for taking care of the energy losses due to friction and heat is by changing the value of γ, as suggested by R. H. Kent. If this is done Dr. J. P. Vinti and Mr. J. Chernick have found that the agreement between this simple theory and more elaborate theories, as judged by the ratio of muzzle energy to maximum pressure, is of the order of one per cent. The maximum pressure, however, may be somewhat more in error, the amount of error depending on the manner of transforming the burning rate coefficient appropriate to a more accurate burning law to a value for use in the first power burning law. If the burning law should be $q_2p^{0.8}$ instead of qp and the correspondence is made by equating qp to $q_2p^{0.8}$ at maximum pressure, the discrepancy in maximum pressure may amount to as much as 30 per cent. On the other hand, if the correspondence is made at $\frac{1}{2}p_{\max}$, the discrepancy may be only a few per cent.

The pressure is related to the free volume v, the amount of powder burnt c, and the velocity of the projectile u by

$$pv = c\lambda - (\gamma - 1)(W/2g)(12u)^2 \tag{5}$$

where $(W/2g)(12u)^2$ is the energy of the projectile, which has all come at the expense of the energy of the propellent gases. Since W includes one-third of the charge weight C, as well as the projectile weight, there is included as "projectile energy" the kinetic energy of motion of the propellent gases. The factor one-third arises from the fact that, if their density distribution down the tube is uniform, the average of the square of the velocity of the gases is one-third the velocity of the projectile squared.

The equation of motion of the projectile is

$$(W/g)(12\dot{u}) = pa. \tag{6}$$

The three equations, (2), (5), and (6) are our fundamental equations. We define

$$p_q = (2qC\lambda/aw)^2(W/gv_0), \tag{7}$$

$$p_c = C\lambda/v_0, \tag{8}$$

both of the dimensions of pressure, the latter being the pressure which would be developed in the chamber if the complete charge burnt before motion of the projectile, that is, with instantaneous combustion.

From (2), since the initial charge burnt is zero, we see that c is proportional to $\int pdt$, whereas from (6) it is seen that u is also proportional to the same integral. The velocity of the projectile is therefore always proportional to the charge burnt up to complete combustion. Using (7) and (8) the relationship can be expressed as

$$c\lambda = \left(\frac{c}{C}\right) p_c v_0 = \sqrt{p_q v_0 (W/g)}(12u) \tag{9}$$

or

$$(W/2g)(12u)^2 = \frac{1}{2}\left(\frac{c}{C}\right)^2 p_c v_0 \left(\frac{p_c}{p_q}\right). \tag{9'}$$

Equation (5) may then be written

$$p(v/v_0) = p_c(c/C)\left[1 - \frac{\gamma - 1}{2}\left(\frac{p_c}{p_q}\right)\left(\frac{c}{C}\right)\right]. \tag{10}$$

We use $du/dt = (du/ds)(ds/dt) = u(du/ds) = 12audu/d(12as) = 12au(du/dv)$, or from (6)

$$p = (W/ga)(12\dot{u}) = (W/g)(12u)d(12u)/dv, \tag{11}$$

and from (9),

$$p = p_c v_0 (p_c/p_q)(c/C)(d/dv)(c/C). \tag{12}$$

With this (10) becomes

$$\frac{d(c/C)}{d \ln v} = \frac{p_q}{p_c} - \frac{\gamma - 1}{2}\frac{c}{C}, \tag{13}$$

$$\begin{aligned} \ln (v/v_0) &= \int_0^{c/C} \frac{dx}{(p_q/p_c) - \frac{1}{2}(\gamma - 1)x} \\ &= -\frac{2}{\gamma - 1} \ln \left[1 - \frac{\gamma - 1}{2}\frac{p_c}{p_q}\frac{c}{C} \right]. \end{aligned} \tag{13'}$$

Then the relation between c/C, the fraction of the charge burnt, and v/v_0, the expansion ratio, is

$$v/v_0 = [1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q)(c/C)]^{-2/(\gamma-1)} \tag{14}$$

$$c/C = [2p_q/p_c(\gamma - 1)][1 - (v/v_0)^{-(\gamma-1)/2}]. \tag{14'}$$

The pressure-expansion ratio equation is, from (10) and (14'):

$$p = [2/(\gamma - 1)]p_q(v/v_0)^{-\gamma}[(v/v_0)^{\frac{1}{2}(\gamma-1)} - 1]. \tag{15}$$

The pressure is a maximum when:

$$v_{\max}/v_0 = [2\gamma/(\gamma + 1)]^{2/(\gamma-1)}, \tag{16}$$

$$c_{\max}/C = p_q/\gamma p_c, \tag{16'}$$

and has the value:

$$p_{\max} = p_q[(\gamma + 1)^{(\gamma+1)}\gamma^{-2\gamma}2^{-(\gamma+1)}]^{1/(\gamma-1)}. \tag{17}$$

The function $f(\gamma) = [(\gamma + 1)^{(\gamma+1)}\gamma^{-2\gamma}2^{-(\gamma+1)}]^{1/(\gamma-1)}$ is $1/e$ for $\gamma = 1$. It does not vary greatly with γ having values:

γ	$f(\gamma)$	γ	$f(\gamma)$
1.0	0.368	1.25	0.310
1.05	0.355	1.30	0.301
1.10	0.342	1.35	0.292
1.15	0.331	1.40	0.283
1.20	0.320		

It is closely approximated by $f(\gamma) \cong e^{-1}[1 + \frac{3}{4}(\gamma - 1)]^{-1}$, the agreement being to at least three significant figures for values of γ to 1.40.

The values of $v_{\max}/v_0$ for different values of γ are:

γ	$v_{\max}/v_0$	γ	$v_{\max}/v_0$
1.0	2.72	1.25	2.32
1.05	2.62	1.30	2.26
1.10	2.54	1.35	2.21
1.15	2.46	1.40	2.16
1.20	2.39		

In terms of the expansion ratio, before all the propellant is consumed, the energy of the projectile, from (9′) and (14′), is:

$$\frac{W}{2g}(12u)^2 = \frac{1}{5.37}Wu^2 = [2p_q v_0/(\gamma - 1)^2][1 - (v/v_0)^{-(\gamma-1)/2}]^2 \quad (18)$$

and the velocity is

$$u = \sqrt{10.74 p_q v_0/W}\,\frac{1}{\gamma - 1}[1 - (v/v_0)^{-(\gamma-1)/2}]. \quad (19)$$

The position of all burnt is given by $c/C = 1$, which from (14) is

$$v_b = v_0[1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q)]^{-2/(\gamma-1)}, \quad (20)$$

and this may be approximated by:

$$v_b = v_0 e^{p_c/p_q}\left\{1 + \frac{1}{4}(\gamma - 1)(p_c/p_q)^2 + \frac{1}{8}(\gamma - 1)^2\left[\frac{2}{3}(p_c/p_q)^3 + \frac{\gamma}{4}(p_c/p_q)^4\right] + \cdots\right\}. \quad (20^*)$$

The pressure at all burnt is:

$$p_b = p_c[1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q)]^{(\gamma+1)/(\gamma-1)} \quad (21)$$

which may be approximated by:

$$p_b = p_c e^{-(p_c/p_q)}[1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q) - \tfrac{1}{4}(\gamma - 1)(p_c/p_q)^2 + \cdots]. \quad (21^*)$$

The energy of the projectile at burnt is:

$$\frac{1}{5.37}Wu_b^2 = \frac{1}{2}C\lambda(p_c/p_q), \quad (22)$$

and the velocity is

$$u_b = \sqrt{2.68(C\lambda/W)(p_c/p_q)}. \quad (22')$$

Beyond the position of burnt the gases obey the equation of adiabatic expansion, pv^γ is constant, so that at the muzzle the pressure is $(v_b/v_m)^\gamma p_b$, and from (21),

$$p_{\text{muzzle}} = p_c(v_0/v_m)^\gamma[1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q)]^{-1}. \quad (23)$$

The muzzle energy is given by:

$$\frac{1}{5.37}Wu_m^2 = \frac{C\lambda}{\gamma - 1}\left\{1 - (v_0/v_m)^{\gamma-1}\left[1 - \frac{1}{2}(\gamma - 1)(p_c/p_q)\right]^{-1}\right\}, \quad (24)$$

* Equations marked by asterisk are approximate expansions which are valid for most weapons.

which may be approximated as:

$$\frac{1}{5.37} W u_m^2 = C\lambda \left\{ \left[\ln (v_m/v_0) - \frac{1}{2} (p_c/p_q) \right] - \frac{1}{2} (\gamma - 1) \left[\ln^2 (v_m/v_0) - (p_c/p_q) \ln (v_m/v_0) + \frac{1}{2} (p_c/p_q)^2 \right] + \cdots \right\}. \quad (24^*)$$

The temperature (absolute) at the muzzle is

$$T_m = T_0(v_0/v_m)^{\gamma-1}[1 - \tfrac{1}{2}(\gamma - 1)(p_c/p_q)]^{-1} \quad (25)$$

where T_0 is the adiabatic flame temperature of the propèllant, that is, the temperature of the uncooled gases produced in the combustion.

II. THE MUZZLE VELOCITY AND MAXIMUM PRESSURE.

Of the equations listed those giving the maximum pressure, (17), and the muzzle energy, (24), are of the greatest practical importance.

The maximum pressure is given by (17) which may be approximated, to far greater accuracy than the theory warrants, by the simple form

$$p_{\max} = (p_q/e)[1 + \tfrac{3}{4}(\gamma - 1)]^{-1} \quad (26)$$

where $e = 2.718$ is the base of the natural logarithm system. Since

$$p_q = (2qC\lambda/aw)^2[W/g(v_c - 17.7C)], \quad (27)$$

it is seen that the maximum pressure is proportional to the projectile weight, depends on the charge as $C^2\Big/\left(1 - \dfrac{17.7C}{v_c}\right)$, depends on burning rate as q^2, on web as w^{-2}, and on the specific force of the propellant as λ^2. The dependence on the chamber volume is as $1/(v_c - 17.7C)$.

Since the burning rate q may be regarded as the least well known of the parameters entering this simplified form of the theory one may use the maximum pressure to determine q, or, more simply, to determine the parameter p_q:

$$p_q = 2.718 p_{\max}[1 + \tfrac{3}{4}(\gamma - 1)]. \quad (28)$$

The equation for the muzzle energy, in inch lbs., may be made to appear simpler by defining the new symbols:

$$r = \ln (v_m/v_0) = 2.303[\log_{10} (12as + v_c - 17.7C) - \log_{10} (v_c - 17.7C)], \quad (29)$$

$$\varphi = p_c/2p_q = (4.02/W)(aw/q)^2(1/C\lambda), \quad (30)$$

$$h = r - \varphi, \quad (31)$$

$$E_m = \frac{1}{5.37} W u_m^2, \quad (32)$$

which latter is the muzzle energy in inch lbs., if the projectile weight, W, is in lbs., and the muzzle velocity, u_m, in feet per second.

In view of (28) φ may be determined from the maximum pressure as

$$\begin{aligned}\varphi &= p_c/\{2ep_{\max}[1 + \tfrac{3}{4}(\gamma - 1)]\} \\ &= C\lambda/\{[v_c - 17.7C]5.436p_{\max}[1 + \tfrac{3}{4}(\gamma - 1)]\}.\end{aligned} \tag{33}$$

This equation is valid only if the point of all burnt comes later than the maximum pressure, which requires that

$$\varphi \gtreqless 1/2\gamma \qquad [\leqq 0.4 \quad \text{if} \quad (\gamma = 1.25)]. \tag{33'}$$

The gun will "spit powder" unless the position of all burnt is before the muzzle. This requires that

$$-\ln[1 - (\gamma - 1)\varphi] \leqq \frac{\gamma - 1}{2} r, \tag{33''}$$

or approximately,

$$\varphi \leqq \tfrac{1}{2}r.$$

The quantity $h = r - \varphi$ is seen to be positive, and equal to or greater than φ, if the gun does not "spit powder."

The equation (24) for the muzzle energy becomes

$$E_m = \frac{C\lambda}{\gamma - 1}\{1 - e^{-(\gamma-1)r}[1 - (\gamma - 1)\varphi]^{-1}\} \tag{34}$$

which may be expanded to the forms

$$\begin{aligned}E_m = C\lambda\Big[&(r - \varphi) - \frac{1}{2}(\gamma - 1)(r^2 - 2r\varphi + 2\varphi^2) \\ &+ \frac{1}{6}(\gamma - 1)^2(r^3 - 3r^2\varphi + 6r\varphi^2 - 6\varphi^3) \\ &- \frac{1}{24}(\gamma - 1)^3(r^4 - 4r^3\varphi + 12r^2\varphi^2 - 24r\varphi^3 + 24\varphi^4)\cdots\Big]\end{aligned} \tag{34*}$$

$$\begin{aligned}E_m = C\lambda\Big[&h - \frac{1}{2}(\gamma - 1)(h^2 + \varphi^2) \\ &+ \frac{1}{6}(\gamma - 1)^2(h^3 + 3h\varphi^2 - 2\varphi^3) \\ &- \frac{1}{24}(\gamma - 1)^3(h^4 + 6h^2\varphi^2 - 8h\varphi^3 + 9\varphi^4)\cdots\Big].\end{aligned} \tag{34**}$$

Except for very long guns the first two terms

$$E_m = C\lambda h\{1 - \tfrac{1}{2}(\gamma - 1)[h + (\varphi^2/h)] + \cdots\} \tag{34***}$$

are sufficient, within an accuracy to which this simple theory applies.

From (34**) it is seen that for guns of moderate length (v_m/v_0 about 20, r about 3 and h about 2), the muzzle energy is proportional to the total "force" of the powder, the product of charge C and the specific force λ, and relatively independent of the value of γ. Since the potential of the powder is $\lambda/(\gamma - 1)$, this quantity for fixed force is highly dependent on γ. That is, the specific force λ, rather than the potential $\lambda/(\gamma - 1)$, represents the capacity of the propellant to do work in the gun, and is a more useful characteristic of the propellant on which to base comparisons of different compositions than the potential.

The variation of muzzle energy E_m, and muzzle velocity u_m, with the various parameters may be most conveniently expressed as logarithmic derivatives, which give the per cent. change in E_m or u_m with unit per cent. change in the parameter. Regarding charge C, projectile weight W, web w, burning constant q, specific heat ratio γ, and specific force λ as parameters, for a given gun (fixed v_c, a, and s) the logarithmic derivatives are as follows:

We use

$$j = \frac{C\lambda}{E_m}\left[1 - (\gamma - 1)\frac{E_m}{C\lambda}\right] = \frac{C\lambda}{E_m} - (\gamma - 1)$$

$$= \frac{1}{h}\left\{1 + \frac{1}{2}(\gamma - 1)[h + (\varphi^2/h) - 2] + \frac{1}{12}(\gamma - 1)^2[h^2 + 4\varphi^3/h + 3\varphi^4/h^2] + \cdots\right\}. \quad (35)$$

This parameter has a value in the neighborhood of 1/2, for most guns.

The per cent. change in muzzle energy for unit per cent. change in projectile weight is, using the conventional designation in thermodynamics of indicating by subscripts to the partial derivatives that web and burning rate are held constant,

$$(\partial \ln E_m/\partial \ln W)_{w,q} = j\varphi[1 - (\gamma - 1)\varphi]^{-1}, \quad (36)$$

and the per cent. change in muzzle velocity is

$$(\partial \ln u_m/\partial \ln W)_{w,q} = \tfrac{1}{2}\{j\varphi[1 - (\gamma - 1)\varphi]^{-1} - 1\}. \quad (36')$$

For all other parameters the per cent. change in muzzle velocity is just half the per cent. change in muzzle energy. The effect of change is

$$\left(\frac{\partial \ln E_m}{\partial \ln C}\right)_{w,q} = 2\frac{\partial \ln u_m}{\partial \ln C}$$

$$= 1 + j\left\{\frac{\varphi}{1 - (\gamma - 1)\varphi} + \frac{17.7C}{v_c}\left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1}\left[1 - \frac{17.7C}{v_c}\right]^{-1}\right\}. \quad (37)$$

The changes due to web, w, and burning rate, q, are equal in magnitude and opposite in sign:

$$\frac{\partial \ln E_m}{\partial \ln q} = -\frac{\partial \ln E_m}{\partial \ln w} = 2\frac{\partial \ln u_m}{\partial \ln q} = -2\frac{\partial \ln u_m}{\partial \ln w} = 2j\varphi[1 - (\gamma - 1)\varphi]^{-1}. \qquad (38)$$

The changes due to force are:

$$\left(\frac{\partial \ln E_m}{\partial \ln \lambda}\right)_{w,q} = 2\frac{\partial \ln u_m}{\partial \ln \lambda} = 1 + j\varphi[1 - (\gamma - 1)\varphi]^{-1}. \qquad (39)$$

If the powder characteristics, projectile weight, and other dimensions of the gun are held constant we have

$$\left(\frac{\partial \ln E_m}{\partial \ln v_c}\right)_{w,q} = 2\frac{\partial \ln u_m}{\partial \ln v_c} = -j\left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1}\left[1 - \frac{17.7C}{v_c}\right]^{-1} \qquad (40)$$

for the effect of the chamber capacity, v_c, and

$$\left(\frac{\partial \ln E_m}{\partial \ln s_m}\right)_{w,q} = 2\frac{\partial \ln u_m}{\partial \ln s_m} = j\left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1} \qquad (41)$$

for the effect of the travel, s_m, from the seating of the projectile to the muzzle.

If, instead of considering the parameters as C, W, w, q, λ and those of the gun, we take C, W, λ, $p_{\max}$ and those of the gun, it is equivalent to assuming that the web, or burning rate, or both, are varied to fix the maximum pressure. In this case we shall indicate the logarithmic derivatives by partials with the subscript p, to indicate that the maximum pressure is held constant. The results are:

$$(\partial \ln E_m/\partial \ln W)_p = 0, \qquad (42)$$

$$(\partial \ln u_m/\partial \ln W)_p = -\frac{1}{2}, \qquad (42')$$

$$\left(\frac{\partial \ln E_m}{\partial \ln C}\right)_p = 2\left(\frac{\partial \ln u_m}{\partial \ln C}\right)_p = 1 + j\left[1 - \frac{17.7C}{v_c}\right]^{-1}\left\{\frac{17.7C}{v_c} \times \left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1} - \frac{\varphi}{1 - (\gamma - 1)\varphi}\right\}, \qquad (43)$$

$$\left(\frac{\partial \ln E_m}{\partial \ln \lambda}\right)_p = 2\left(\frac{\partial \ln u_m}{\partial \ln \lambda}\right)_p = 1 - j\varphi[1 - (\gamma - 1)\varphi]^{-1}. \qquad (44)$$

Similarly if web or burning rate or both are adjusted to maintain the same maximum pressure the effects of varying the chamber volume, v_c or travel, s_m are:

$$\left(\frac{\partial \ln E_m}{\partial \ln v_c}\right)_p = 2\left(\frac{\partial \ln u_m}{\partial \ln v_c}\right)_p = j\left[1 - \frac{17.7C}{v_c}\right]^{-1}\left\{\frac{\varphi}{1 - (\gamma - 1)\varphi} - \left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1}\right\}, \quad (45)$$

$$\left(\frac{\partial \ln E_m}{\partial \ln s_m}\right)_p = 2\left(\frac{\partial \ln u_m}{\partial \ln s_m}\right)_p = j\left[1 + \frac{v_c - 17.7C}{12as_m}\right]^{-1}. \quad (46)$$

Finally the change of muzzle energy with maximum pressure, if obtained by changing web or burning rate only, is

$$\frac{\partial \ln E_m}{\partial \ln p_{\max}} = 2\frac{\partial \ln u_m}{\partial \ln p_{\max}} = j\varphi[1 - (\gamma - 1)\varphi]^{-1}. \quad (47)$$